

# Four-Dimensional Quantum Hall Effect in a Two-Dimensional Quasicrystal

Yaacov E. Kraus, Zohar Ringel, and Oded Zilberberg

*Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel*

One-dimensional (1D) quasicrystals exhibit physical phenomena associated with the 2D integer quantum Hall effect. Here we transcend dimensions, and show that a previously inaccessible phase of matter – the 4D integer quantum Hall effect – can be incorporated in a 2D quasicrystal. Correspondingly, our 2D model has a quantized charge-pump accommodated by an elaborate edge phenomena with protected level crossings. We propose experiments to observe these 4D phenomena, and generalize our results to a plethora of topologically equivalent quasicrystals. Thus, 2D quasicrystals may pave the way to the experimental study of 4D physics.

PACS numbers: 71.23.Ft, 73.43.-f, 73.43.Nq, 05.30.Rt

The three-dimensionality of our world has never prevented physicists from studying their theories in arbitrary dimensions. In recent decades this has proven useful in mesoscopic physics, as many systems are well described by two, one and even zero dimensions. However, dimensions higher than three, which host many fascinating phenomena, are seemingly out of reach.

The uprising field of topological phases of matter is also dealing with systems of arbitrary dimension [1, 2]. In this paradigm, each energy gap of a system is attributed an index, which is robust to continuous deformations. A nontrivial index is usually associated with interesting boundary phenomena, quantized response, and exotic quasiparticles. While nontrivial topological phases appear in any dimension [3, 4], the physical manifestations are limited to 1D [5, 6], 2D [7, 8] and 3D [9].

An example for an intriguing topological phase, which is seemingly out of reach, is the 4D generalization of the 2D integer quantum Hall effect (IQHE). In 2D, a uniform magnetic field creates Landau levels that are characterized by the 1<sup>st</sup> Chern number – the topological index that corresponds to the quantized Hall conductance [10, 11]. In 4D, a uniform SU(2) Yang-Mills field results in generalized Landau levels [12, 13]. These levels are characterized by the 2<sup>nd</sup> Chern number – a topological index which corresponds to a quantized non-linear response [13, 14]. Both the 2D and 4D IQHEs exhibit a variety of exotic strongly correlated phases when interactions are included [13, 15]. Hence, the 4D IQHE, as well as lattice models with non-vanishing 2<sup>nd</sup> Chern numbers, constantly attract theoretical attention [16, 17].

Recently, it was shown that 1D quasicrystals (QCs) – non-periodic structures with long-range order – exhibit topological properties of the 2D IQHE [18]. The bulk energy spectrum of these 1D QCs is gapped, and each gap is associated with a nontrivial 1<sup>st</sup> Chern number. This association relies on the fact that the long-range order harbors an additional degree-of-freedom in the form of a shift of the quasiperiodic order. Accordingly, boundary states traverse the gaps as a function of this shift. This property was observed in photonic QCs, and was utilized for an adiabatic pump of light [18]. Moreover, upon a

deformation between QCs with different 1<sup>st</sup> Chern numbers, a phase transition is expected to occur. Such a transition was also observed experimentally in photonic QCs [19]. Generalizations to other 1D symmetry classes, physical implementations, and QCs were discussed, see e.g. Refs. [20–23].

In this work, we take a major step further, and present a 2D QC that exhibits topological properties of the 4D IQHE. Each gap in its energy spectrum is characterized by a nontrivial 2<sup>nd</sup> Chern number, which implies: (i) quantized charge pumping with an underlying 4D symmetry; (ii) gap-traversing edge states with protected level crossings; (iii) quantum phase transitions between topologically distinct QCs. Generalizations to other models and other types of QCs are discussed. We propose two experiments to measure the 2<sup>nd</sup> Chern number via charge pumping, and, thus, make 4D physics experimentally accessible.

We study a 2D tight-binding model of spin- $\frac{1}{2}$  particles that hop on a square lattice in the presence of a modulated on-site potential

$$H(\phi_x, \phi_y) = \sum_{x,y} \sum_{\sigma=\pm} c_{x,y,\sigma}^\dagger \left[ t_x c_{x+1,y,\sigma} + t_y c_{x,y+1,\sigma} + \text{h.c.} + (l_x \cos(\sigma 2\pi b_x x + \phi_x) + l_y \cos(\sigma 2\pi b_y y + \phi_y)) c_{x,y,\sigma} \right]. \quad (1)$$

Here  $c_{x,y,\sigma}$  is the single-particle annihilation operator of a particle at site  $(x, y)$  with spin  $\sigma$ ;  $t_x, t_y$  are the hopping amplitudes in the  $x$ - and  $y$ -directions; and  $l_x, l_y$  are the amplitudes of the on-site potentials, which are modulated along  $x$  and  $y$  with modulation frequencies  $b_x, b_y$ , respectively [cf. Fig. 1(a)]. Lastly, the Hamiltonian depends on two shift parameters  $\phi_x$  and  $\phi_y$ . We assume the modulation frequencies  $b_x$  and  $b_y$  to be irrational, which makes the on-site modulations incommensurate with the lattice, and the model becomes quasiperiodic.

The spectrum of  $H$  is gapped. Therefore, it may exhibit nontrivial topological indices. Note that  $H$  is neither symmetric to time reversal nor to charge conjugation. Therefore, according to conventional topological classifications [3, 4], the only apparent topological index that can be associated with it is the 1<sup>st</sup> Chern number.

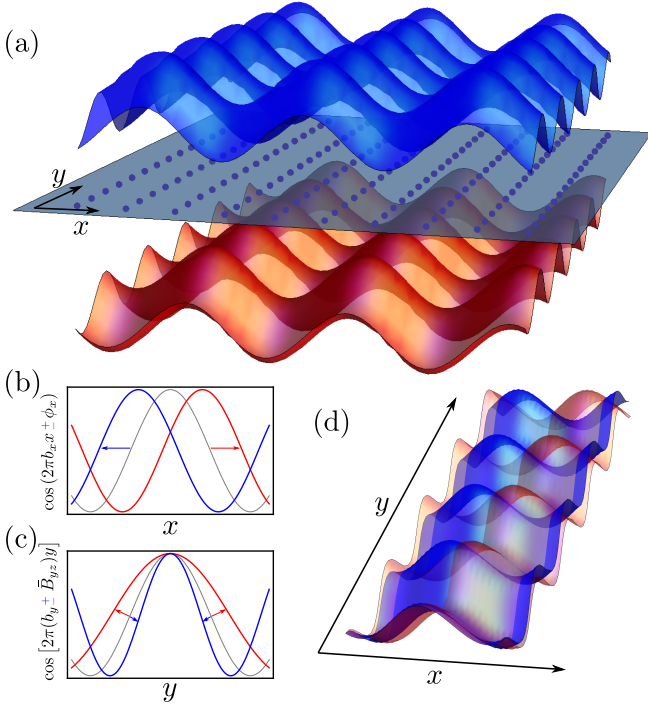


FIG. 1. (a) An illustration of  $H$  [cf. Eq. (1)]: a 2D square lattice with a modulated on-site potential that is spin dependent. The spin-up state,  $\sigma = +$ , experiences the upper (blue) potential, and the spin-down state,  $\sigma = -$ , experiences the lower (red). For clarity, the potentials are vertically-displaced. The dots mark the underlying lattice sites. (b)-(d) A quantized charge-pumping along the  $x$ -direction is achieved by scanning the shift parameters  $\phi_x$  and  $\phi_y$  in the presence of the modulation modifications  $\bar{B}_{yz}$  and  $\bar{B}_{wy}$ , respectively [cf. Eqs. (5) and (6)]. The effects on the potentials of  $\sigma = +$  (blue) and  $\sigma = -$  (red) is illustrated (the thin (gray) line denotes the unmodified reference potential): (b)  $\phi_x$  (or equivalently  $\phi_y$ ) shifts the cosine-potentials in opposite directions for opposite spins. (c)  $\bar{B}_{yz}$  makes the modulation frequency become spin dependent  $b_y + \sigma \bar{B}_{yz}$ . (d)  $\bar{B}_{wy}$  shifts the cosine-potentials in opposite directions, but with an increasing shift along the  $y$ -direction.

However, for  $\phi_x = \phi_y = 0$ ,  $H$  is, in fact, time-reversal symmetric, and any  $H(\phi_x, \phi_y)$  is adiabatically connected to  $H(0, 0)$ . Hence, the 1<sup>st</sup> Chern number of  $H$  vanishes, and  $H$  is seemingly trivial.

Strikingly,  $H$  is attributed a nontrivial 2<sup>nd</sup> Chern number, which, by definition, characterizes 4D systems. In order to obtain this result, let us consider  $H$  on a toroidal geometry [24]. We introduce twisted boundary conditions along the  $x$  and  $y$  directions, parameterized by  $\theta_x$  and  $\theta_y$ , respectively. For a given gap and given  $\phi_\mu \equiv (\phi_x, \phi_y, \theta_x, \theta_y)$ , we denote by  $P(\phi_\mu)$  the projection matrix on all the eigenstates of  $H(\phi_\mu)$  with energies below this gap. We can now define,

$$\mathcal{C}(\phi_\mu) = \sum_{\alpha\beta\gamma\delta} \frac{\epsilon_{\alpha\beta\gamma\delta}}{-8\pi^2} \text{Tr} \left( P \frac{\partial P}{\partial \phi_\alpha} \frac{\partial}{\partial \phi_\beta} P \frac{\partial P}{\partial \phi_\gamma} \frac{\partial P}{\partial \phi_\delta} \right), \quad (2)$$

where  $\epsilon_{\alpha\beta\gamma\delta}$  is the antisymmetric tensor of rank-4.

Formally, the 2<sup>nd</sup> Chern number is defined by  $\mathcal{V} = \int d^4\phi_\mu \mathcal{C}(\phi_\mu)$  [25]. By its definition,  $\mathcal{V}$  characterizes a 4D family of Hamiltonians composed of all  $H(\phi_\mu)$  with all possible values of  $\phi_\mu$ , i.e.  $\phi_x$  and  $\phi_y$  act as two additional effective dimensions. The main result of this Letter is that even for a given  $\phi_\mu$ , the gaps of  $H(\phi_\mu)$  can be associated with a nontrivial integer 2<sup>nd</sup> Chern number,

$$\mathcal{V} = (2\pi)^4 \mathcal{C}(\phi_\mu) \neq 0, \quad (3)$$

in the thermodynamic limit. Below, we justify Eq. (3) by showing that, for our model,  $\mathcal{C}(\phi_\mu)$  is essentially independent of  $\phi_\mu$ , and thus the integration over the four parameters is redundant.

Beforehand, we present the physical implications of the nontrivial  $\mathcal{V}$ . To do so, we apply to  $H$  the procedure of *dimensional extension* that was introduced in Refs. [18, 21]. In this procedure, we interpret  $\phi_x$  and  $\phi_y$  as momenta along two fictitious perpendicular coordinates  $w$  and  $z$ , respectively. Now, the Hamiltonian  $H(\phi_x, \phi_y)$  is a single Fourier component of some ancestor 4D Hamiltonian  $\mathcal{H}$ . By making the inverse Fourier transform, we obtain a Hamiltonian describing spin- $\frac{1}{2}$  particles hopping on a 4D hyper-cubic lattice

$$\mathcal{H} = \sum_{\mathbf{x}, \hat{\mu}} \mathbf{c}_{\mathbf{x}}^\dagger e^{i2\pi a_{\hat{\mu}}(\mathbf{x})} t_{\hat{\mu}} \mathbf{c}_{\mathbf{x}+\hat{\mu}} + \text{h.c.}, \quad (4)$$

where  $\mathbf{c}_{\mathbf{x}} = (c_{\mathbf{x},+}, c_{\mathbf{x},-})$  annihilates a spin- $\frac{1}{2}$  particle at site  $\mathbf{x} = (x, y, z, w)$ ,  $\hat{\mu}$  is summed over the unit vectors  $\hat{x}, \hat{y}, \hat{z}$  and  $\hat{w}$ , which connect nearest neighbors, and  $t_{\hat{\mu}} = (t_x, t_y, \lambda_x/2, \lambda_y/2)$ . The particles are coupled to a Yang-Mills gauge field  $a_{\hat{\mu}}(\mathbf{x}) = (b_y y \hat{z} + b_x x \hat{w}) \sigma_3$ . This vector potential describes a spin-polarized uniform SU(2) field. Such a field is known to generate a 4D IQHE with a nontrivial  $\mathcal{V}$  [12, 13]. Notably,  $\mathcal{H}$  is defined on a planar geometry in a Landau gauge, whereas previous analyses treated a spherical geometry in a symmetric gauge.

Similar to the 2D IQHE,  $\mathcal{V}$  has a physical manifestation in the form of a response function. Here the response is quantized but non-linear:  $j_\alpha = \mathcal{V} \frac{e^2}{h\Phi_0} \epsilon_{\alpha\beta\gamma\delta} B_{\beta\gamma} E_\delta$  [14], where  $j_\alpha$  denotes the current density along the  $\alpha$ -direction,  $\Phi_0$  is the flux quantum,  $E_\delta$  is an electric field along the  $\delta$ -direction, and  $B_{\beta\gamma}$  is a magnetic field in the  $\beta\gamma$ -plane. Note that  $B_{\beta\gamma}$  couples to the electric charge, and affects equally both spin components.

A direct observation of this response requires a 4D system. However, we can develop an analogue of Laughlin's pumping, which is manifested in the 2D QC. Let us consider the following two cases:  $j_x = \mathcal{V} \frac{e^2}{h\Phi_0} B_{yz} E_w$  and  $j_x = \mathcal{V} \frac{e^2}{h\Phi_0} B_{wy} E_z$ . Recall that the electric fields,  $E_w$  and  $E_z$ , can be generated by time-dependent Aharonov-Bohm fluxes,  $E_w = \frac{1}{caN_w} \partial_t \Phi_w(t)$  and  $E_z = \frac{1}{caN_z} \partial_t \Phi_z(t)$ , where  $a$  is the lattice spacing, and  $N_w$  and  $N_z$  are the number of lattice sites along the  $w$ - and  $z$ -directions, respectively. Expressing  $B_{yz}$  and  $B_{wy}$  in  $\mathcal{H}$  in Landau

gauge, and performing dimensional reduction, these fields enter the 2D model,  $H$ , through modified on-site terms,

$$\lambda_x \cos[2\pi(\sigma b_x x + \bar{B}_{wy}y) + \phi_x(t)] + \lambda_y \cos[2\pi(\sigma b_y y + \bar{B}_{yz}y) + \phi_y(t)], \quad (5)$$

where  $\phi_x(t) = \frac{2\pi}{N_w\Phi_0}\Phi_w(t)$  and  $\phi_y(t) = \frac{2\pi}{N_z\Phi_0}\Phi_z(t)$ , and  $\bar{B}_{wy} = B_{wy}a^2/\Phi_0$  and  $\bar{B}_{yz} = B_{yz}a^2/\Phi_0$  denote the corresponding flux quanta per plaquette. Figures 1(b)-(d) illustrate the effects of these modifications. Now, fixing the chemical potential within a gap with a given  $\mathcal{V}$ , an adiabatic scan of  $\phi_x$  or of  $\phi_y$  from 0 to  $2\pi$  pumps charge along the  $x$ -direction, such that

$$Q_x = \mathcal{V}e\bar{B}_{yz}N_y, \quad (6a)$$

$$Q_x = \mathcal{V}e\bar{B}_{wy}N_y, \quad (6b)$$

respectively, where  $N_y$  is the number of lattice sites along the  $y$ -direction.

We can now propose experiments that measure  $\mathcal{V}$  using Eq. (6). For that purpose, take a 2D slab of the QC and connect metal leads to the edges of the  $x$ -coordinate. Let us assume that the chemical potential of both the QC and the leads is placed in some gap of  $H$ . Then, one should measure the charge-flow during the scan of  $\phi_x$  from 0 to  $2\pi$  for different values of  $\bar{B}_{yz}$  [26]. According to Eq. (6a), we expect that  $\mathcal{V} = \frac{1}{eL_y}\partial Q_x/\partial \bar{B}_{yz}$ . Similarly, according to Eq. (6b), during the scan of  $\phi_y$  while varying  $\bar{B}_{wy}$ , charge flows and  $\mathcal{V} = \frac{1}{eL_y}\partial Q_x/\partial \bar{B}_{wy}$ . Remarkably, due to the 4D origin of our model, the measured  $\mathcal{V}$  would be the same in both experiments.

We have just seen that upon a scan of  $\phi_x$ , charge may flow in the  $x$ -direction. Consequently, for an open geometry, in order to accommodate this charge transfer, edge states must appear and traverse the gaps as a function of  $\phi_x$ . These states appear for infinitesimally small  $\bar{B}_{yz}$ , and hence appear also for  $\bar{B}_{yz} = 0$ . Figure 2 depicts the numerically obtained energy spectrum of  $H$  as a function of  $\phi_x$ , for an open  $x$ -coordinate, a periodic  $y$ -coordinate,  $t_x = t_y = 1$ ,  $l_x = l_y = 1.8$ ,  $x = y = 1.34$ ,  $b_x = (1+\sqrt{5})/2$  and  $b_y = 55/34$ , which approximates  $(1+\sqrt{5})/2$  to order  $10^{-4}$  [24]. The spectrum is invariant with respect to  $\phi_y$ , and is depicted for  $\phi_y = 0$ . As a function of  $\phi_x$ , the spectrum has flat bands and gap-traversing bands. The flat bands correspond to bulk states, whereas the gap-traversing ones to edge state (see insets). The edge states are divided into four types: spin-up and spin-down states (blue and red), which are localized at either the left or right edge (opposite slopes). These edge states are a signature of the nontrivial  $\mathcal{V}$  of our model. They can be measured in a way similar to the experiments performed in 1D photonic QCs [18].

Naively, one could postulate that opposite-spin modes that reside on the same edge could be gapped out by introducing spin-mixing terms. However, this is not the

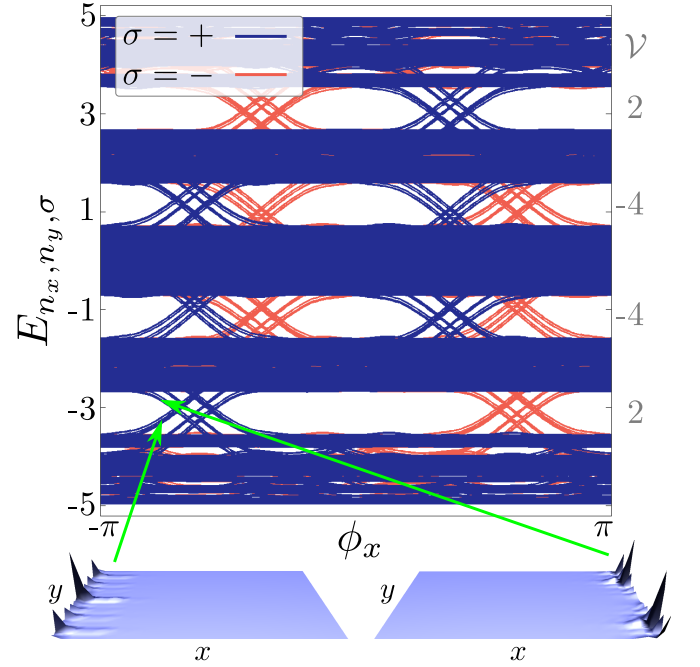


FIG. 2. The spectrum of  $H$  as a function of the shift parameter  $\phi_x$  [cf. Eq. (1)], for an open  $x$ -coordinate and a periodic  $y$ -coordinate. The horizontal bands correspond to bulk states. The gap-traversing states with  $\sigma = +$  (blue) or  $\sigma = -$  (red) are edge states, which are localized at the left or right edges (see insets for typical wave functions). The crossings of  $\sigma = +$  and  $\sigma = -$  bands are topologically protected. The values of the 2<sup>nd</sup> Chern number,  $\mathcal{V}$ , associated with the large gaps are presented.

case, and the edge modes and their crossings are topologically protected. In order to establish this protection, we decompose  $H$  into its spin and spatial constituents. The Hamiltonian  $H$  is a sum of two spin-components, where each spin-component is subject to two decoupled 1D Harper models along the  $x$ - and  $y$ -directions. Both spin components experience the same modulation frequencies,  $b_x$  and  $b_y$ , but couple to the shift parameters,  $\phi_x$  and  $\phi_y$ , with an opposite sign. Therefore, each eigenstate of  $H(\phi_x, \phi_y)$  is a product of eigenstates of the Harper models in the  $x$ - and  $y$ -directions and a spin eigenstate.

Recall that each gap of the Harper model is associated with a nontrivial 1<sup>st</sup> Chern number, which corresponds to the number of boundary states that traverse the gap as a function of  $\phi$  [10, 11, 18]. Accordingly, each band of the quasicrystalline Harper models in the  $x$ - and  $y$ -directions is associated with some spin-dependent Chern number,  $\nu_{r_x, \sigma}$  and  $\nu_{r_y, \sigma}$ , where  $r_x$  and  $r_y$  denote the corresponding bands, respectively [27]. In Fig. 2, the bands that traverse the gaps as a function of  $\phi_x$  are composed of products of bulk bands in the  $y$ -direction and boundary states in the  $x$ -direction. Notably, due to the opposite coupling of the spin to  $\phi_x$  and  $\phi_y$ , the Chern numbers of bands

of opposite spin have opposite signs,  $\nu_{r_x,+} = -\nu_{r_x,-}$  and  $\nu_{r_y,+} = -\nu_{r_y,-}$ . Therefore, the gaps are traversed by the same number of spin-up and spin-down bands, but with opposite slopes. Since opposite-spin bands are associated with opposite 1<sup>st</sup> Chern numbers, they cannot be gapped-out by spin-mixing terms, even if they cross at some value of  $\phi_x$ . Otherwise,  $\nu_{r_y,\sigma}$  would change continuously as a function of  $\phi_x$  between  $\nu_{r_y,+}$  and  $\nu_{r_y,-}$ . The level crossing is therefore protected.

The described edge phenomena accounts for the charge-pumping appearing in Eq. (6). When  $\phi_x$  or  $\phi_y$  are scanned, spin-up and spin-down states, which have opposite 1<sup>st</sup> Chern numbers, flow in opposite directions. In the absence of  $\bar{B}_{yz}$  and  $\bar{B}_{wy}$ , the two charge currents cancel. Applying  $\bar{B}_{yz}$  or  $\bar{B}_{wy}$  causes a difference between the densities of spin-up and spin-down states, and thus a net charge is pumped [14]. This corresponds to the idea of associating the 2<sup>nd</sup> Chern number with pumping of 1<sup>st</sup> Chern numbers [28]. Remarkably, even for vanishing  $\bar{B}_{yz}$  and  $\bar{B}_{wy}$ , spin is pumped across the sample. Therefore, for recursive scans of  $\phi_x$ , macroscopic spins accumulate at the boundaries.

We turn, now, to establish Eq. (3). Let us evoke the definition of the 1<sup>st</sup> Chern number of a band of the Harper model in the  $x$ -direction with spin  $\sigma$ ,  $\nu_{r_x,\sigma} = \int d\phi_x d\theta_x C_{r_x,\sigma}(\phi_x, \theta_x)$ , where  $C_{r_x,\sigma}(\phi_x, \theta_x) = \frac{1}{2\pi i} \text{Tr}(P_{r_x,\sigma}[\partial_{\phi_x} P_{r_x,\sigma}, \partial_{\theta_x} P_{r_x,\sigma}])$ , and  $P_{r_x,\sigma}(\phi_x, \theta_x)$  is the projection matrix on the eigenstates of the  $r_x$ th band [25]. A similar definition applies for  $\nu_{r_y,\sigma}$ . Let us denote by  $\epsilon_n(\phi)$  the eigenenergies of the Harper model. The decomposition of  $H$  into  $x$ - and  $y$ -constituents makes its energy spectrum a Minkowski sum,  $E_{n_x, n_y, \sigma}(\phi_x, \phi_y) = \epsilon_{n_x}(\sigma\phi_x) + \epsilon_{n_y}(\sigma\phi_y)$ . Accordingly, the states below each gap of  $E_{n_x, n_y, \sigma}$  can be decomposed into a sum over pairs of bands in the 1D spectra,  $r_x$  and  $r_y$ , such that  $\epsilon_{r_x} + \epsilon_{r_y} < \mu$ , where  $\mu$  is the energy in the middle of the gap. Using the fact that the eigenfunctions of  $H$  are a product of the 1D eigenfunctions, we obtain [29]

$$\mathcal{V} = \sum_{\sigma=\pm} \sum_{(\epsilon_{r_x} + \epsilon_{r_y} < \mu)} \nu_{r_x,\sigma} \nu_{r_y,\sigma} \neq 0. \quad (7)$$

In a previous work [18], we have shown that, in the thermodynamic limit,  $C(\phi, \theta)$  becomes independent of  $\phi$  and  $\theta$ . Hence  $\nu_{r_x,\sigma} = (2\pi)^2 C_{r_x,\sigma}(\phi_x, \theta_x)$  and  $\nu_{r_y,\sigma} = (2\pi)^2 C_{r_y,\sigma}(\phi_y, \theta_y)$ . This, combined with Eq. (7), immediately implies Eq. (3) [29].

The invariance of  $C(\phi, \theta)$  of the Harper model with respect to  $\phi$  follows from the correspondence of  $\phi$  to lattice translations. The modulated potential  $\cos(2\pi b x + \phi)$  implies that a translation by  $n$  lattice sites is equivalent to a shift of  $\phi$  by  $2\pi(bn \bmod 1)$ . Provided that  $b$  is irrational, the set of  $\phi$ -shifts generated by all lattice translations is dense in the interval  $[0, 2\pi]$ , in the thermodynamic limit. Consequently, any shift of  $\phi$  is equivalent, to arbitrary precision, to some lattice translation, which has no effect

on  $C(\phi, \theta)$ . Turning back to  $H$ , the invariance of  $\mathcal{C}(\phi_\mu)$  with respect to  $\phi_x$  and  $\phi_y$  results from the equivalence of their shifts to lattice translations in  $x$  and  $y$ , respectively, where the spin components are translated in opposite directions.

Until now, our analysis used the spin-flip symmetry of  $H$ :  $\sigma \rightarrow -\sigma$ . Let us break this symmetry by making the modulation frequencies spin-dependent, i.e.  $b_x \rightarrow b_{x,\sigma}$  and  $b_y \rightarrow b_{y,\sigma}$ . Now, a translation by  $n$  lattice sites, for example in the  $x$ -direction, is equivalent to spin-dependent shifts of  $\phi_x$  by  $2\pi(\sigma b_{x,\sigma} n \bmod 1)$ . Ostensibly,  $\phi_x$  is no longer equivalent to lattice translations. Nonetheless, if  $b_{x,+}$  and  $b_{x,-}$  are mutually irrational, the Kronecker-Weyl theorem [30] implies that the space of spin-dependent shifts generated by lattice translations along  $x$  is dense over  $[0, 2\pi]^2$ . In particular, the line created by shifts of  $\phi_x$  is also densely covered. Consequently, here too, any shift of  $\phi_x$  is equivalent, to arbitrary precision, to some lattice translation. The same argumentation applies in the  $y$ -direction. Thus, Eq. (3) holds also in the absence of spin-flip symmetry.

The above symmetry breaking may result from a U(1) gauge field in the 4D Hamiltonian  $\mathcal{H}$ . In fact, any  $\text{SU}(2) \times \text{U}(1)$  gauge transformation in 4D that respects Landau gauge keeps the system unchanged. After the dimensional reduction to 2D, such a transformation becomes a general local transformation that may mix  $x$ ,  $y$ , and spin, but keeps  $\mathcal{V}$  unchanged. More generally, the properties of  $\mathcal{C}(\phi_\mu)$  implies that any unitary transformation of the 2D Hamiltonian that does not depend on  $\phi_\mu$  keeps  $\mathcal{C}(\phi_\mu)$  independent of  $\phi_\mu$ .

The predicted 2<sup>nd</sup> Chern number is not limited to a 2D QC composed of two Harper models. One can consider (i) placing the cosine modulations in the hopping terms, rather than in the on-site terms (off-diagonal Harper), and (ii) replacing each of the cosine modulations with a Fibonacci-like modulation. In 1D, all these variants were shown to be topologically equivalent to the Harper model, namely they have the same distribution of 1<sup>st</sup> Chern numbers [21]. Therefore, the gaps in such 2D QCs variants will have nontrivial  $\mathcal{V}$ .

To conclude, in this Letter we have presented a novel 2D quasicrystalline model that is associated with the same topological index as the 4D IQHE – the 2<sup>nd</sup> Chern number. Correspondingly, our model exhibits an elaborate edge phenomena and a quantized charge-pump. We propose an experiment in which the charge is pumped through the system following a modification of the quasiperiodic modulation. Interestingly, while these modifications differ considerably, they lead to the same pumped charge. This equivalence may seem baffling from a 2D perspective, but follows from a simple symmetry of the corresponding non-linear response in 4D. Recent progress in controlling and engineering systems, such as optical lattices [31] and photonic crystals [32], makes our 2D model seem experimentally feasible. Moreover, in-

interactions in such systems may lead to fractional quasicrystalline phases which are descendants of the exotic 4D fractional quantum Hall effect [13]. Thus, our model serves as a porthole by which to access 4D physics.

We thank E. Berg for fruitful discussions. We acknowledge the Minerva Foundation of the DFG and the US-Israel Binational Science Foundation for financial support. Authors' names appear in alphabetical order.

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## SUPPLEMENTAL MATERIAL

In the main text, we notice that the Hamiltonian  $H$  [cf. Eq. (1) of the main text] can be decomposed into Harper models in the  $x$ - and  $y$ -directions, for both spin species. Consequently, we infer that the 2<sup>nd</sup> Chern numbers  $\mathcal{V}$  of  $H$  can be expressed as multiplications of the 1<sup>st</sup> Chern numbers of the constituting Harper models [cf. Eq. (7) of the main text]. In this Supplemental Material we prove this preposition.

Recall that for a given gap in the spectrum of  $H(\phi_\mu)$ , we define the projection matrix  $P(\phi_\mu)$  on all the eigenstates of  $H(\phi_\mu)$  with energies below this gap. Similar to  $H$ ,  $P = \sum_{\sigma=\pm} P_\sigma$ , where  $P_\sigma$  projects on the  $\sigma$ -spin states, and thus  $P_+P_- = 0$ . Additionally, we note that the states below the gap of the 2D model,  $H$ , are composed of pairs of 1D bands that satisfy  $\epsilon_{r_x} + \epsilon_{r_y} < \mu$ , where  $\epsilon_r$  is an eigenenergy of the  $r$ th band of the Harper model, and  $\mu$  is an energy in the middle of the 2D gap. Therefore,

$$P(\phi_\mu) = \sum_{\sigma=\pm} \sum_{(\epsilon_{r_x} + \epsilon_{r_y} < \mu)} P_{r_x,\sigma}(\phi_x, \theta_x) \otimes P_{r_y,\sigma}(\phi_y, \theta_y), \quad (\text{I.1})$$

where  $P_{r_x,\sigma}(\phi_x, \theta_x)$  is the projection matrix on the eigenstates of the  $r_x$ th band, and similarly for  $P_{r_y,\sigma}(\phi_y, \theta_y)$ . Note that the elements of this sum are orthogonal projectors.

In the main text, we introduced the 2<sup>nd</sup> Chern form,  $\mathcal{C}(\phi_\mu)$  [cf. Eq. (2) of the main text], and the 2<sup>nd</sup> Chern number,  $\mathcal{V} = \int d^4\phi_\mu \mathcal{C}(\phi_\mu)$ . Since  $\mathcal{V}$  is a  $\mathbb{Z}$ -index, a 2<sup>nd</sup> Chern number can also be assigned to each element of the sum in Eq. (I.1). It suffices, therefore, to focus on a single such element, and show that its 2<sup>nd</sup> Chern number is a product of its corresponding 1<sup>st</sup> Chern numbers.

Let us denote such an element by  $\bar{P}(\phi_\mu) = P_{r_x,\sigma}(\phi_x, \theta_x) \otimes P_{r_y,\sigma}(\phi_y, \theta_y)$ , and its corresponding 2<sup>nd</sup> Chern form by  $\bar{\mathcal{C}}(\phi_\mu)$ . For brevity, we denote  $\partial\bar{P}/\partial\phi_\alpha$  by  $\bar{P}^{[\alpha]}$ , which makes  $\bar{\mathcal{C}}(\phi_\mu) = \epsilon_{\alpha\beta\gamma\delta} \text{Tr}(\bar{P}\bar{P}^{[\alpha]}\bar{P}^{[\beta]}\bar{P}^{[\gamma]}\bar{P}^{[\delta]})/(-8\pi^2)$ . According to the decomposition of  $\bar{P}$  into  $x$ - and  $y$ -parts,  $\bar{\mathcal{C}}$  involves two types of terms:

- (i)  $\text{Tr}(P_{r_x,\sigma}P_{r_x,\sigma}^{[\alpha]}P_{r_x,\sigma}P_{r_x,\sigma}^{[\gamma]}P_{r_x,\sigma} \otimes P_{r_y,\sigma}P_{r_y,\sigma}^{[\beta]}P_{r_y,\sigma}P_{r_y,\sigma}^{[\delta]}P_{r_y,\sigma})$ ,
- (ii)  $\text{Tr}(P_{r_x,\sigma}P_{r_x,\sigma}^{[\alpha]}P_{r_x,\sigma}^{[\beta]}P_{r_x,\sigma} \otimes P_{r_y,\sigma}P_{r_y,\sigma}^{[\gamma]}P_{r_y,\sigma}^{[\delta]}P_{r_y,\sigma})$ .

The type-(i) terms vanish, since for any projector,  $P$ , the following applies:  $PP^{[\alpha]}P = 0$ , where  $P^{[\alpha]}$  is a derivative of  $P$  with respect to some variable  $\alpha$ . Recall that for any two matrices,  $\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$ . This makes the type-(ii) terms become  $\text{Tr}(P_{r_x,\sigma}P_{r_x,\sigma}^{[\alpha]}P_{r_x,\sigma}^{[\beta]})\text{Tr}(P_{r_y,\sigma}P_{r_y,\sigma}^{[\gamma]}P_{r_y,\sigma}^{[\delta]})$ . Substituting this into  $\bar{\mathcal{C}}$ , we obtain

$$\begin{aligned} \bar{\mathcal{C}}(\phi_\mu) &= \frac{\epsilon_{\alpha\beta}}{2\pi i} \text{Tr}(P_{r_x,\sigma}P_{r_x,\sigma}^{[\alpha]}P_{r_x,\sigma}^{[\beta]}) \cdot \frac{\epsilon_{\gamma\delta}}{2\pi i} \text{Tr}(P_{r_y,\sigma}P_{r_y,\sigma}^{[\gamma]}P_{r_y,\sigma}^{[\delta]}) \\ &\equiv C_{r_x,\sigma}(\phi_x, \theta_x) C_{r_y,\sigma}(\phi_y, \theta_y), \end{aligned} \quad (\text{I.2})$$

where  $\epsilon_{\alpha\beta}$  is the antisymmetric tensor of rank-2, and  $C(\phi, \theta)$  is the 1<sup>st</sup> Chern form. The 2<sup>nd</sup> Chern number corresponding to  $\bar{\mathcal{C}}$  is

$$\begin{aligned} \bar{\mathcal{V}} &= \int d^4\phi_\mu \bar{\mathcal{C}}(\phi_\mu) \\ &= \int d\phi_x d\theta_x C_{r_x,\sigma} \cdot \int d\phi_y d\theta_y C_{r_y,\sigma} = \nu_{r_x,\sigma} \nu_{r_y,\sigma}, \end{aligned} \quad (\text{I.3})$$

where  $\nu_{r_x,\sigma}$  and  $\nu_{r_y,\sigma}$  are the 1<sup>st</sup> Chern numbers associated with  $P_{r_x,\sigma}(\phi_x, \theta_x)$  and  $P_{r_y,\sigma}(\phi_y, \theta_y)$ , respectively.

$\bar{\mathcal{V}}$  is the 2<sup>nd</sup> Chern number of a single term in the sum (I.1), i.e. with given  $r_x$ ,  $r_y$  and  $\sigma$ . Summing over all terms, we obtain the total 2<sup>nd</sup> Chern number,  $\mathcal{V}$ , given by Eq. (7) of the main text.